Indian Statistical Institute
B.Math. (Hons.) III Year
First Semester Back Paper Exam 2006-07
Introduction to Differential Geometry

Time: 3 hrs Date: -01-07
Instructor: Maneesh Thakur

Attempt all questions, they carry equal marks. You may use any result proved in the course.

1. Let $\gamma(t)$ be a unit speed curve in $\mathbb{R}^3$ with curvature $k(t) \neq 0$ for all t. Define a new curve δ by
$$\delta(t) = \frac{d\gamma(t)}{dt}.$$
Show that δ is regular. If s is the arc length parameter for δ, show that
$$\frac{ds}{dt} = k$$
and the curvature of δ is $\sqrt{(1 + \frac{\tau^2}{k^2})}$ where τ is the torsion of γ.

2. Let $\gamma(t)$ be a simple closed curve of period a. Let $\mathbf{t}, \mathbf{n}_s$ & $\mathbf{k}_s$ denote its unit tangent vector, signed unit normal and signed curvature respectively. Show that
$\mathbf{t}(t+a) = \mathbf{t}(t),\ \mathbf{n}_s(t+a) = \mathbf{n}_s(\mathbf{t})$ and $\mathbf{k}_s(t+a) = \mathbf{k}_s(t)$.

3. Let σ be the surface patch $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ for a surface of revolution, where $u \mapsto (f(u), 0, g(u))$ is a unit speed curve in $\mathbb{R}^3$. Compute the geodetic curvature of
(i) a meridian $v=$ constant
(ii) a parallel u=constant.

4. Let σ be a surface patch with $\mathbf{N}$ as its standard unit normal. Let
$$F_3 = \begin{pmatrix} \|\mathbf{N}_u\|^2 & \mathbf{N}_u \cdot \mathbf{N}_v \\ \mathbf{N}_u \cdot \mathbf{N}_v & \|\mathbf{N}_v\|^2 \end{pmatrix}.$$

Let F_1 and F_2 denote the matrices of the first and second fundamental forms of σ respectively. Show that $F_3 = F_2 F_1^{-1} F_2$.

5. Let H and K denote the mean and Gaussian curvatures of a surface patch σ. Let F_1, F_2, F_3 be as in (4). Show that

$$F_3 - 2HF_2 + KF_1 = 0.$$

6. For a geodetic triangle ABC on the pseudosphere, show that the area $\mathcal{A}(ABC)$ is given by

$$\mathcal{A}(ABC) = \pi - \angle A - \angle B - \angle C.$$